Simultaneous Quadratic Stabilization of a Nonholonomic Mobile Robot

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***Abstract*— Nonholonomic wheeled mobile robots (WMRs) are finding more and more applications in the present world. Varieties of applications demand precise and accurate control of the robot. To achieve accurate tracking as well as asymptotic stability, a control scheme based on Simultaneous Quadratic Stabilization (SQS) has been proposed. A reduced order system dynamics is formulated eliminating the associated constraints to apply the proposed control scheme. The actuator dynamics have also been incorporated to achieve precise motion control. The method probably proposed first time for nonholonomic wheeled mobile robotic system shows prospective results as verified through simulation.**

Keywords—wheeled mobile robot; nonholonomic system dynamics; simultaneous quadratic stabilization.

I. INTRODUCTION

It has been a long time that researchers are working in the field of nonholonomic systems to achieve effective controllers for accurate performance. Differentially driven wheeled mobile robot represents a class of nonholonomic system that is used in variety of applications e.g., planetary exploration, inspection, service sector, etc. Nonholonomic constraints, which are non-integrable, do not reduce the dimensionality of the configuration space whereas a holonomic constraint does. For nonholonomic constraints every configuration can be reached from any other configuration, although that may not be achieved in an arbitrary manner. These properties pose major challenge for controlling the nonholonomic systems like wheeled mobile robot.

The controllable degrees of freedom of this class of systems are less than their global degrees of freedom. Thus, these systems require more number of variables than their mobility to define their configuration. The configuration of the robot depends upon the application it is being used to. Different control schemes have been used over time to achieve better control of WMRs like feedback linearization [1], sliding mode control [8], robust control [6], adaptive control [9], etc. Dynamic modeling of mobile robots having nonholonomic constraints and control of the same using the concept of feedback linearization applying the tools of Lie Algebra had been studied earlier [1,13]. Some of the robust wheel mobile robot controllers based on neural network and sliding mode approaches have been presented in [11,12] which also deal with uncertainty.

In this paper, a solitary stabilizing feedback gain matrix based on SQS method is employed to ensure asymptotic tracking while following a desired trajectory. Once, the nonlinear system is expressed as a set of Linear Time Invariant (LTI) systems and the state equations are in controllable companion form, SQS can be used to find the controller parameters and a common quadratic Lyapunov function for the overall system.

The rest of the paper is organized as follows: section II presents the formulation of the dynamic equation in desired form; section III presents the theory and the algorithm used; section IV presents the simulation parameters and results obtained; finally conclusion is drawn in section V.

II. DYNAMIC FORMULATION

A. Dynamic Model of a Constrained Mechanical System

The dynamic model of the mechanical system in an n-dimensional configuration space $\mathbf{C}$ with generalized coordinates $\mathbf{q}=(q_1,.....,q_n)$ subject to m (assuming $m<n$) constraints is considered. The equations of motion of a wheeled mobile robot (WMR) moving on a horizontal plane & associated with constraints are described in [1,14] as:

$$\mathbf{M(q)\ddot{q}+V(q,\dot{q})=B(q)\tau-A^T(q)\lambda}, \tag{1}$$

$$\mathbf{A(q)\dot{q}=0}, \tag{2}$$

where, $\mathbf{M(q)}\in\mathbb{R}^{n\times n}$ is the symmetric, positive definite inertia matrix, $\mathbf{V(q,\dot{q})=C(q,\dot{q})\dot{q}}$ (where $\mathbf{C(q,\dot{q})}\in\mathbb{R}^{n\times n}$) is the centrifugal and coriolis forces matrix, $\mathbf{B(q)}\in\mathbb{R}^{n\times(n-m)}$ is the input transformation matrix, $\boldsymbol{\tau}\in\mathbb{R}^{(n-m)\times 1}$ is the external input vector, $\mathbf{A}^T\mathbf{(q)}\in\mathbb{R}^{n\times m}$ is the transpose of the constraints matrix, $\boldsymbol{\lambda}\in\mathbb{R}^{m\times 1}$ is the vector of constraint forces (Lagrange multipliers). The kinematic model of the system is given as,

$$\mathbf{\dot{q}=S(q)v(t)}, \tag{3}$$

where $\mathbf{S(q)}\in\mathbb{R}^{n\times(n\text{-}m)}$ is a full rank matrix made up with the vector fields i.e. $\mathbf{S(q)}=\left[\mathbf{s}_1\mathbf{(q)},.......,\mathbf{s}_{n-m}\mathbf{(q)}\right]$so that,

$$\mathbf{A(q)S(q)=0} \text{ or } \mathbf{S^T(q)A^T(q)=0} \tag{4}$$

and $\mathbf{v(t)=[v_1(t),.......,v_{n\text{-}m}(t)]^T}$, are also called pseudo velocities. Differentiating (3), we obtain,

$$\ddot{\mathbf{q}} = \mathbf{S}(\mathbf{q})\dot{\mathbf{v}}(\mathrm{t}) + \dot{\mathbf{S}}(\mathbf{q})\mathbf{v}(\mathrm{t}). \tag{5}$$

B. *Consideration for the wheeled mobile robot*

In this endeavor, a wheeled mobile robot with two active wheels which are differentially driven and a passive castor wheel which is used to provide stability is considered. The wheels are driven by Brushless DC motors associated with gearboxes. The robot has been assumed to have a rigid frame and non-deformable wheels. A few assumptions have been made such as; the wheels are of same radius and the centre of mass (COM) lies on the axis of symmetry. The schematic diagram of the WMR is given in Fig. 1.

C. *Nomenclature of the robot*

$\{\mathrm{W}:\mathrm{O},\mathrm{X},\mathrm{Y}\}$ - The fixed coordinate frame. $\{\mathrm{R}:\mathrm{O}_1,\mathrm{X}_1,\mathrm{Y}_1\}$ - The body fixed or moving coordinate frame. $\mathrm{S}(\mathrm{x_c},\mathrm{y_c})$ - COM of the whole system. $\mathrm{P}(\mathrm{x_p},\mathrm{y_p})$ - COM of the platform. $\mathrm{W_L}(\mathrm{x_L},\mathrm{y_L})$ - COM of the left wheel and adjoining rotary parts. $\mathrm{W_R}(\mathrm{x_R},\mathrm{y_R})$ - COM of the right wheel and adjoining rotary parts. L - Length of the robot. 2b - Distance between the two wheels up to the contact points. r - Radius of both the driving wheels. d - Distance between the centre of mass S and the point O_1. Δ - The offset between point S and P along $\mathrm{O_1X_1}$. m_p - Mass of the robot platform. m_w - Mass of each wheel, its driving motor and the associated gearbox. I_p - Moment of inertia (MOI) of the platform about a vertical axis passing through point P. I_{m} - MOI of each wheel, its driving motor and the associated gearbox about a vertical axis passing through its COM. I_w - Equivalent MOI of wheel referred to the its axis.

D. *Equations of Motion*

The dynamic equations of a mobile robot are derived from the Euler-Lagrange equation of motion. For a wheeled mobile robot, the Euler-Lagrange equations of motion for *n*-number of generalized coordinates are given as,

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{\Gamma} - \mathbf{A}^{\mathrm{T}}(\mathbf{q})\boldsymbol{\lambda} \quad , \quad \text{subjected to} \quad \mathbf{A}(\mathbf{q})\dot{\mathbf{q}} = \mathbf{0} \quad ,$$

where, $\mathbf{\Gamma}$ is the vector of generalized forces, L is the Lagrangian of the system which is defined as, $L(\mathbf{q},\dot{\mathbf{q}}) =$ Kinetic Energy-Potential Energy. The pose of the mobile robot in global frame is specified by the coordinate of the COM of the robot $(\mathrm{x_c},\mathrm{y_c})$ and the heading angle φ. Heading angle φ is the orientation of the body fixed frame with respect to the global frame. The rotation of the right and the left wheel is denoted by θ_{r} and θ_{l} respectively and the torque in the right and left wheel is denoted as τ_{r} & τ_{l} respectively. The detailed dynamics of the wheel mobile robot considered here are given in [1] as:

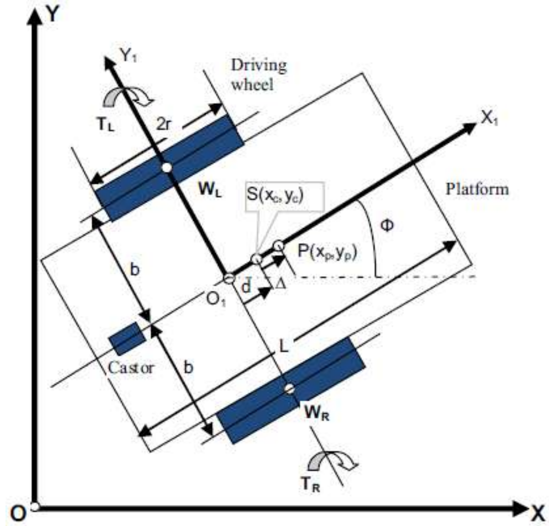

Fig. 1. Schematic Diagram of the wheeled mobile robot

$$m\ddot{x}_c + \mathrm{K_r}(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) - \lambda_1\sin\varphi - \cos\varphi(\lambda_2 + \lambda_3) = 0, \tag{6}$$

$$m\ddot{y}_c - \mathrm{K_r}(\ddot{\varphi}\cos\varphi - \dot{\varphi}^2\sin\varphi) + \lambda_1\cos\varphi - \sin\varphi(\lambda_2 + \lambda_3) = 0, \tag{7}$$

$$I\ddot{\varphi} + \mathrm{K_r}(\ddot{x}_c\sin\varphi - \ddot{y}_c\cos\varphi) - \mathrm{d}\lambda_1 + \mathrm{b}(\lambda_3 - \lambda_2) = 0, \tag{8}$$

$$I_w\ddot{\theta}_{\mathrm{r}} + \lambda_2 r = \tau_{\mathrm{r}} \tag{9}$$

$$I_w\ddot{\theta}_{\mathrm{l}} + \lambda_3 r = \tau_{\mathrm{l}} \quad , \tag{10}$$

where, $m = m_p + 2m_w$, $\mathrm{K_r} = (2m_w d - m_p\Delta)$, & $I = I_p + m_p\Delta^2 + 2m_w(b^2 + d^2) + 2I_{\mathrm{m}}$.

E. *Reduced order dynamical model of WMR*

For a WMR, the actual controllable parameters are the position, velocity and acceleration of the wheels i.e. the motors driving them. Here, intention is to obtain a reduced order dynamical form only in terms of the wheel parameters $(\theta_{\mathrm{r}}\ \&\ \theta_{\mathrm{l}})$ and its higher derivatives planning a reference trajectory in terms of the global pose $(\mathrm{x_c},\mathrm{y_c},\varphi)$. In this endeavor, equations (6), (7) and (8), are solved for the constraint forces which are given as,

$$\lambda_1 = m(\ddot{x}_c\sin\varphi - \ddot{y}_c\cos\varphi) + \mathrm{K_r}\ddot{\varphi} \tag{11}$$

$$\lambda_2 = \frac{(mb\cos\varphi + \mathrm{K_r}\sin\varphi - dm\sin\varphi)}{2b}\ddot{x}_c + \frac{(mb\sin\varphi - \mathrm{K_r}\cos\varphi + dm\cos\varphi)}{2b}\ddot{y}_c + \frac{(I - \mathrm{K_r}\,d)}{2b}\ddot{\varphi} + \frac{\mathrm{K_r}}{2}\dot{\varphi}^2 \tag{12}$$

$$\lambda_3 = \frac{(mb\cos\varphi - \mathrm{K_r}\sin\varphi + dm\sin\varphi)}{2b}\ddot{x}_c + \frac{(mb\sin\varphi + \mathrm{K_r}\cos\varphi - dm\cos\varphi)}{2b}\ddot{y}_c - \frac{(I - \mathrm{K_r}\,d)}{2b}\ddot{\varphi} + \frac{\mathrm{K_r}}{2}\dot{\varphi}^2 \tag{13}$$

Substituting (12) and (13) in (9) and (10) respectively, and we obtain,

$$I_w\ddot{\theta}_{\mathrm{r}} + \Big(\tfrac{(mb\cos\varphi + \mathrm{K_r}\sin\varphi - dm\sin\varphi)}{2b}\ddot{x}_c + \tfrac{(mb\sin\varphi - \mathrm{K_r}\cos\varphi + dm\cos\varphi)}{2b}\ddot{y}_c + \tfrac{(I - \mathrm{K_r}\,d)}{2b}\ddot{\varphi} + \tfrac{\mathrm{K_r}}{2}\dot{\varphi}^2\Big) r = \tau_{\mathrm{r}} \tag{14}$$

$$I_w\ddot{\theta}_l+\Big(\tfrac{(mb\cos\varphi-K_r\sin\varphi+dm\sin\varphi)}{2b}\ddot{x}_c+\tfrac{(mb\sin\varphi+K_r\cos\varphi-dm\cos\varphi)}{2b}\ddot{y}_c-\tfrac{(I-K_r d)}{2b}\ddot{\varphi}+\tfrac{K_r}{2}\dot{\varphi}^2\Big)r=\tau_l \quad (15)$$

The kinematic model of the WMR has already been given as, $\dot{\mathbf{q}}=\mathbf{S}(\mathbf{q})\mathbf{v}(t)$. For the mobile robot the generalized coordinate vector, $\mathbf{q}=\begin{bmatrix}x_c & y_c & \varphi & \theta_r & \theta_l\end{bmatrix}^T$ and the velocity vector, $\mathbf{v}(t)=\begin{bmatrix}\dot{\theta}_r & \dot{\theta}_l\end{bmatrix}^T$ are considered, where $\mathbf{S}(\mathbf{q})$ is given by:

$$\mathbf{S(q)}=\begin{bmatrix}\frac{r}{2b}(b\cos\varphi-d\sin\varphi) & \frac{r}{2b}(b\cos\varphi+d\sin\varphi)\\ \frac{r}{2b}(b\sin\varphi+d\cos\varphi) & \frac{r}{2b}(b\sin\varphi-d\cos\varphi)\\ \frac{r}{2b} & -\frac{r}{2b}\\ 1 & 0\\ 0 & 1\end{bmatrix}.$$

Partitioning the $\mathbf{S}(\mathbf{q})$ matrix and the generalized coordinate vector $\mathbf{q}$ in the following form:

$\mathbf{S}(\mathbf{q})=\begin{bmatrix}\mathbf{S}_1(\mathbf{q})\\ \mathbf{S}_2(\mathbf{q})\end{bmatrix}$ & $\mathbf{q}=\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{bmatrix}$, we can write:

$$\dot{\mathbf{q}}=\begin{bmatrix}\dot{\mathbf{q}}_1\\ \dot{\mathbf{q}}_2\end{bmatrix}=\begin{bmatrix}\mathbf{S}_1(\mathbf{q})\\ \mathbf{S}_2(\mathbf{q})\end{bmatrix}\mathbf{v}(t),$$

with,

$$\mathbf{S}_1(\mathbf{q})=\begin{bmatrix}\frac{r}{2b}(b\cos\varphi-d\sin\varphi) & \frac{r}{2b}(b\cos\varphi+d\sin\varphi)\\ \frac{r}{2b}(b\sin\varphi+d\cos\varphi) & \frac{r}{2b}(b\sin\varphi-d\cos\varphi)\\ \frac{r}{2b} & -\frac{r}{2b}\end{bmatrix},$$

$\mathbf{q}_1=\begin{bmatrix}x_c & y_c & \varphi\end{bmatrix}^T$ & $\mathbf{q}_2=\begin{bmatrix}\theta_r & \theta_l\end{bmatrix}^T$. So, we can also say, $\dot{\mathbf{q}}_2=\mathbf{v}(t)$. Thus the reduced order kinematic equation can be expressed compactly as:

$$\dot{\mathbf{q}}_1=\mathbf{S}_1(\mathbf{q})\dot{\mathbf{q}}_2. \quad (16)$$

The dynamics represented through (14) and (15) require to know the parameters like $\ddot{x}_c$, $\ddot{y}_c$, $\ddot{\varphi}$ and $\dot{\varphi}$. Differentiating the modified kinematic equation, we obtain expressions of $\ddot{x}_c$, $\ddot{y}_c$, $\ddot{\varphi}$ and we already have the expression of $\dot{\varphi}$ from (16). Substituting these expressions in (14) and (15), the modified dynamics entails the form:

$$\begin{bmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_r\\ \ddot{\theta}_l\end{bmatrix}+\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}\tau_r\\ \tau_l\end{bmatrix}, \quad (17)$$

where,

$$M_{11}=M_{22}=I_w+Ih^2+mh^2(b^2+d^2)-2K_r h^2 d,$$
$$M_{12}=M_{21}=-Ih^2+mh^2(b^2-d^2)+2K_r h^2 d,$$
$$C_1=\tfrac{K_r}{2}rh^3(\dot{\theta}_r-\dot{\theta}_l)^3-K_r h^3 b(\dot{\theta}_r^2-\dot{\theta}_l^2)+2mh^3bd\,\dot{\theta}_l(\dot{\theta}_r-\dot{\theta}_l),$$
$$C_2=\tfrac{K_r}{2}rh^3(\dot{\theta}_r-\dot{\theta}_l)^3+K_r h^3 b(\dot{\theta}_r^2-\dot{\theta}_l^2)-2mh^3bd\,\dot{\theta}_r(\dot{\theta}_r-\dot{\theta}_l),$$

with, $h=\dfrac{r}{2b}$.

F. Associated Actuator Dynamics

For driving the WMR considered here, it is assumed that two actuators (Brushless DC motor) are interfaced with the wheels through gear mechanism. A motor is used to convert electrical energy into mechanical energy while driving the mechanical system. The dynamic equation for the motor neglecting the inductance is given in [1] as,

$$\boldsymbol{\tau}=K_{m1}\mathbf{V}_m-K_{m2}\mathbf{v}, \quad (18)$$

where, $\boldsymbol{\tau}=[\tau_r \quad \tau_l]^T$ is the motor torque vector, $\mathbf{V}_m=\begin{bmatrix}V_{mr} & V_{ml}\end{bmatrix}^T$ is the motor control voltage vector. K_{m1} and K_{m2} are constants whose expressions are given as, $K_{m1}=\dfrac{NK_T}{R_a}$ and $K_{m2}=\dfrac{N^2K_TK_b}{R_a}$, where N is the speed reduction ration and R_a is the resistance of the motor winding. The torque provided by the motor, is directly proportional to the current, i.e. $\tau_m=K_T\, i_m$, where, τ_m is the torque generated by the motor, i_m is the current to the motor and K_T is the motor torque constant. The back EMF being generated at the armature, is directly proportional to the velocity of the motor shaft, i.e. $E_b=K_b\,\omega_m$, where, E_b is the back EMF being generated, ω_m is the velocity of the motor shaft and K_b is the velocity constant.

Substituting (18) in (17), we finally obtain the reduced order robot dynamics in the following form:

$$\begin{bmatrix}M_1 & M_2\\ M_3 & M_4\end{bmatrix}\begin{bmatrix}\ddot{\theta}_r\\ \ddot{\theta}_l\end{bmatrix}+\begin{bmatrix}C_1\\ C_2\end{bmatrix}=\begin{bmatrix}V_{mr}\\ V_{ml}\end{bmatrix}\Rightarrow \mathbf{M}_R\ddot{\mathbf{q}}_2+\mathbf{C}_R(\dot{\mathbf{q}}_2)=\mathbf{V}_m, \quad (19)$$

where,

$$M_1=M_4=(I_w+Ih^2+mh^2(b^2+d^2)-2K_r h^2 d)/K_{m1},$$
$$M_2=M_3=(-Ih^2+mh^2(b^2-d^2)+2K_r h^2 d)/K_{m1},$$
$$C_1=\Big(\tfrac{K_r}{2}rh^3(\dot{\theta}_r-\dot{\theta}_l)^3-K_r h^3 b(\dot{\theta}_r^2-\dot{\theta}_l^2)+2mh^3bd\,\dot{\theta}_l(\dot{\theta}_r-\dot{\theta}_l)+K_{m2}\dot{\theta}_r\Big)/K_{m1},$$
$$C_2=\Big(\tfrac{K_r}{2}rh^3(\dot{\theta}_r-\dot{\theta}_l)^3+K_r h^3 b(\dot{\theta}_r^2-\dot{\theta}_l^2)-2mh^3bd\,\dot{\theta}_r(\dot{\theta}_r-\dot{\theta}_l)+K_{m2}\dot{\theta}_l\Big)/K_{m1}..$$

III. Controller Design

Based on the reduced order dynamic formulation presented in section-II, SQS [2] method was used for controller design to achieve accurate tracking and asymptotic stability of the mobile robotic system. The prerequisite of applying SQS is that, the system should be in controllable canonical form. The detailed control scheme is explained in the subsequent sections.

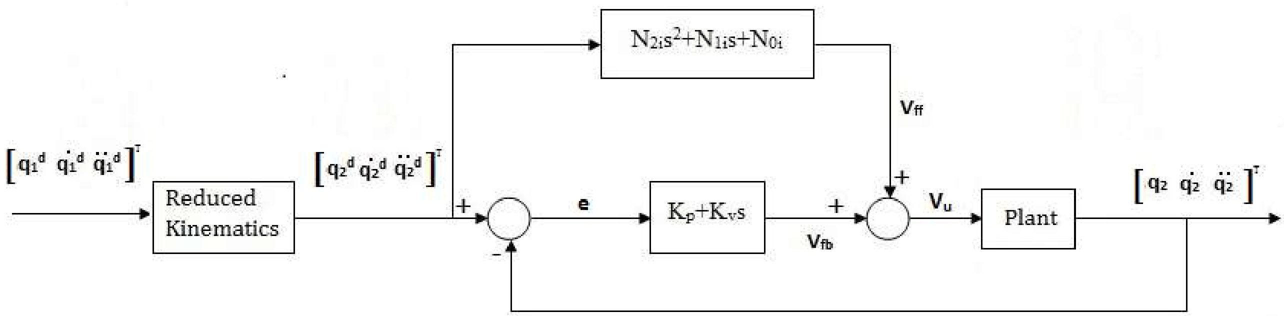

Fig. 2. Block Diagram representation of Simultaneous Quadratic Stabilization based Robot Controller

A. Simultaneous Quadratic Stabilization

The SQS method is applicable for a class of LTI systems described by the state equation of the form:

$$\dot{\mathbf{x}} = \mathbf{A}_i\mathbf{x} + \mathbf{Bu} \tag{20}$$

where, $\mathbf{A}_i \in \mathbb{R}^{n\times n}$ & $\mathbf{B} \in \mathbb{R}^{n\times m}$. This set of systems represented in (20) is Simultaneously Quadratically Stabilizable, if there exists a symmetric positive definite matrix $\mathbf{W}$ such that, $\mathbf{W} = \mathbf{W}^{\mathrm{T}} > \mathbf{0}$ and $\mathbf{K}$ such that [2],

$$\mathbf{x}^{\mathrm{T}}((\mathbf{A}_i - \mathbf{BK})^{\mathrm{T}}\mathbf{W}^{-1} + \mathbf{W}^{-1}(\mathbf{A}_i - \mathbf{BK}))\mathbf{x} < \mathbf{0},$$

$$\forall \mathbf{x} \in \mathbb{R}^n \text{ and } \forall\, i = 0,1,\cdots,\mathrm{n}.$$

Further, if the above inequality holds true, then we have a common quadratic Lyapunov function $\bar{V}(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{x}$ and a control law of the form $\mathbf{u} = -\mathbf{Kx}$ where, $\mathbf{K}$ is a single static state feedback gain matrix such that the time derivative of the Lyapunov function $\bar{V}(\mathbf{x})$ is negative definite along the trajectory. A necessary and sufficient condition for SQS method can be deduced from [4] and [5], which is given as the existence of a positive definite matrix $\mathbf{W}$ such that,

$$\mathbf{x}^{\mathrm{T}}(\mathbf{W}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{W})\mathbf{x} < \mathbf{0} \tag{21}$$

where, $\mathbf{x}$ belongs to the null space of $\mathbf{B}^{\mathrm{T}}$. The control law is as follows:

$$\mathbf{u} = -\mathbf{Kx} = -\tfrac{\gamma}{2}\mathbf{B}^{\mathrm{T}}\mathbf{W}^{-1}\mathbf{x} \tag{22}$$

where, γ is a large positive scalar. γ is chosen as, $\gamma \geq max\{\gamma_i\}$ where γ_i is obtained from,

$$\mathbf{W}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{W} - \gamma_i\mathbf{B}\mathbf{B}^{\mathrm{T}} < \mathbf{0} \tag{23}$$

If there is uncertainty in the input matrix as well, the input matrix is considered in the form, $\mathbf{B}_i = \mathbf{B}\mathbf{E}_i$ with $\mathbf{E}_i = \mathbf{L}_i\mathbf{L}_i^{\mathrm{T}} > \mathbf{0}$, where $\mathbf{L}_i$ is a non-singular matrix. The SQS problem for a system with uncertain input matrix is same as that without uncertainty but the required controller gain will be different as the computation of γ_i changes. A positive scalar γ_i exists if the following inequality holds true,

$$\mathbf{W}\mathbf{A}_i^{\mathrm{T}} + \mathbf{A}_i\mathbf{W} - \gamma_i\mathbf{B}\mathbf{L}_i\mathbf{L}_i^{\mathrm{T}}\mathbf{B}^{\mathrm{T}} < \mathbf{0} \tag{24}$$

which can be shown using Finsler's Lemma.

B. Linearisation of the reduced dynamics of WMR

A linearized model of the reduced order robot dynamics represented by (19) requires to be obtained about each nominal point over the specified trajectory to achieve the form as given in (20). As the inertia matrix of the dynamic equation given in (19) doesn't change, the system described by (19) can be linearized to obtain the following form [3],

$$\mathbf{N}_{2i}\ddot{\mathbf{q}}_2 + \mathbf{N}_{1i}\dot{\mathbf{q}}_2 + \mathbf{N}_{0i}\mathbf{q}_2 = \mathbf{V}_u, \tag{25}$$

where, $\mathbf{N}_{2i}$, $\mathbf{N}_{1i}$, $\mathbf{N}_{0i}$ are (2×2) matrices expressed by:

$$\mathbf{N}_{2i} = \left[\mathbf{M}_{\mathbf{R}}(\mathbf{q}_2)\right]_{S^i},\quad \mathbf{N}_{1i} = \left[\frac{\partial \mathbf{C}_{\mathbf{R}}}{\partial \dot{\mathbf{q}}_2}\right]_{S^i},\quad \mathbf{N}_{0i} = \left[\frac{\partial \mathbf{C}_{\mathrm{R}}}{\partial \mathbf{q}_2}\right]_{S^i},\ \text{with}$$

$S^i = (\mathbf{q}_{2i}, \dot{\mathbf{q}}_{2i}, \ddot{\mathbf{q}}_{2i})$, which denotes the i^{th} nominal point on the reference trajectory. Denoting, $\mathbf{q}_2 = \mathbf{q}_3$ & $\dot{\mathbf{q}}_2 = \dot{\mathbf{q}}_3 = \mathbf{q}_4$, to represent the linearized system (25) in state space form and finally the state space representation is given by:

$$\frac{\mathrm{d}}{\mathrm{dt}}\begin{bmatrix}\mathbf{q}_3\\ \mathbf{q}_4\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{2\times2} & \mathbf{I}_{2\times2}\\ -\mathbf{N}_{2i}^{-1}\mathbf{N}_{0i} & -\mathbf{N}_{2i}^{-1}\mathbf{N}_{1i}\end{bmatrix}\begin{bmatrix}\mathbf{q}_3\\ \mathbf{q}_4\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{2\times2}\\ \mathbf{N}_{2i}^{-1}\end{bmatrix}\mathbf{V}_u, \tag{26}$$

describing the structure of the $\mathbf{A}_i$ & $\mathbf{B}_i$ matrix as,

$\mathbf{A}_i = \begin{bmatrix}\mathbf{A}_{11} & \mathbf{A}_{12}\\ \mathbf{A}_{21}^i & \mathbf{A}_{22}^i\end{bmatrix}$ and $\mathbf{B}_i = \begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix}$, with $\mathbf{A}_{11} = \mathbf{0}_{2\times2}$, $\mathbf{A}_{12} = \mathbf{I}_2$, $\mathbf{A}_{21}^i = -\mathbf{N}_{2i}^{-1}\mathbf{N}_{0i}$, $\mathbf{A}_{22}^i = -\mathbf{N}_{2i}^{-1}\mathbf{N}_{1i}$, $\mathbf{B}_1 = \mathbf{0}_{2\times2}$ and $\mathbf{B}_2 = \mathbf{N}_{2i}^{-1}$.

C. The Algorithm used

- We have the values of the matrices $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$. We choose a matrix $\mathbf{F}$ such that the matrix $\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{F}$ is stable, i.e. its eigenvalues have negative real parts.
- Now we find out $\mathbf{W}_1 = \mathbf{W}_1^{\mathrm{T}} > \mathbf{0}$ by solving,

  $(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{F})\mathbf{W}_1 + \mathbf{W}_1(\mathbf{A}_{11} + \mathbf{A}_{12}\mathbf{F})^{\mathrm{T}} = \mathbf{Q}_0$, where $\mathbf{Q}_0$ is a 2×2 dimensional negative definite matrix.
- Now we compute $\mathbf{W}_2$ as, $\mathbf{W}_2 = (\mathbf{F}\mathbf{W}_1)^{\mathrm{T}}$.

- The matrix $\mathbf{W}=\begin{bmatrix}\mathbf{W}_1 & \mathbf{W}_2\\ \mathbf{W}_2^{\mathrm{T}} & \mathbf{W}_3\end{bmatrix}$ must be a symmetric positive definite matrix. Thus to ensure that it is so, $\mathbf{W}_3$ is chosen as, $\mathbf{W}_3=\mathbf{R}+\mathbf{W}_2^{\mathrm{T}}\mathbf{W}_1^{-1}\mathbf{W}_2$, where $\mathbf{R}$ is any positive definite matrix.
- Now we need to find out the value of the scalar γ. It is computed as,

$$\gamma_i\mathbf{I}_m > (\mathbf{Q}_{2i}-\mathbf{Q}_i^{\mathrm{T}}\mathbf{Q}_0^{-1}\mathbf{Q}_i)\mathbf{E}_i^{-1},$$

where, $\mathbf{Q}_i=\mathbf{A}_{11}\mathbf{W}_2+\mathbf{A}_{12}\mathbf{W}_3+\mathbf{W}_1(\mathbf{A}_{21}^i)^{\mathrm{T}}+\mathbf{W}_2\mathbf{A}_{22}^i$

and $\mathbf{Q}_{2i}=\mathbf{A}_{21}^i\mathbf{W}_2+\mathbf{A}_{22}^i\mathbf{W}_3+\mathbf{W}_2^{\mathrm{T}}(\mathbf{A}_{21}^i)^{\mathrm{T}}+\mathbf{W}_3\mathbf{A}_{22}^i$.

- The control law is deduced based on the above values and given by,

$$\mathbf{K}=\frac{\gamma}{2}\mathbf{B}^{\mathrm{T}}\mathbf{W}^{-1}.$$

D. Controller Design

Here, the control scheme is based on the linearized robot dynamics obtained in (26). The control scheme is represented in Fig. 2. The controller is a combination of feedforward and feedback entities. The total control law is given as,

$$\mathbf{V}_u(t)=\mathbf{V}_{\mathrm{ff}}(t)+\mathbf{V}_{\mathrm{fb}}(t) \quad . \tag{27}$$

The linearized model of the robot is used to compute the feedforward controller, which provides tracking ability. The controller can be written as,

$$\mathbf{V}_{\mathrm{ff}}(t)=\mathbf{N}_{2i}\ddot{\mathbf{q}}_2^{\mathrm{d}}+\mathbf{N}_{1i}\dot{\mathbf{q}}_2^{\mathrm{d}}+\mathbf{N}_{0i}\mathbf{q}_2^{\mathrm{d}} \tag{28}$$

where, $\mathbf{q}_2^{\mathrm{d}}$ represents the reference wheel position. Thus, the feedforward controller is activated from the reference trajectory and can be used to achieve tracking at each nominal point on the reference trajectory. The feedforward affects the steady state response of the tracking error.

On the other hand, the feedback controller is employed to achieve asymptotic stability of the whole system. If we substitute the expression of $\mathbf{V}_{\mathrm{ff}}(t)$, i.e. (28) in (27), we obtain the error dynamics as:

$$\mathbf{N}_{2i}\ddot{\mathbf{q}}_2+\mathbf{N}_{1i}\dot{\mathbf{q}}_2+\mathbf{N}_{0i}\mathbf{q}_2=\mathbf{N}_{2i}\ddot{\mathbf{q}}_2^{\mathrm{d}}+\mathbf{N}_{1i}\dot{\mathbf{q}}_2^{\mathrm{d}}+\mathbf{N}_{0i}\mathbf{q}_2^{\mathrm{d}}+\mathbf{V}_{\mathrm{fb}}(t)$$

$$\Rightarrow \mathbf{N}_{2i}\ddot{\mathbf{e}}+\mathbf{N}_{1i}\dot{\mathbf{e}}+\mathbf{N}_{0i}\mathbf{e}=-\mathbf{V}_{\mathrm{fb}}(t), \tag{29}$$

where, $\mathbf{e}(t)=\left(\mathbf{q}_2^{\mathrm{d}}-\mathbf{q}_2\right)$ gives the tracking error. It can also be written in a state space form as,

$$\dot{\boldsymbol{\eta}}(t)=\mathbf{A}_i\boldsymbol{\eta}(t)-\mathbf{B}_i\mathbf{V}_{\mathrm{fb}}(t) \tag{30}$$

where, $\boldsymbol{\eta}(t)=\left[\mathbf{e}(t)^{\mathrm{T}} \quad \dot{\mathbf{e}}(t)^{\mathrm{T}}\right]^{\mathrm{T}}$. Implementing a state feedback controller of the form,

$$\mathbf{V}_{\mathrm{fb}}(t)=-\mathbf{K}\boldsymbol{\eta}(t)=-\mathbf{K}_{\mathrm{p}}\mathbf{e}(t)-\mathbf{K}_{\mathrm{v}}\dot{\mathbf{e}}(t),$$

where, $\mathbf{K}=\left[\mathbf{K}_{\mathrm{p}} \quad \mathbf{K}_{\mathrm{v}}\right]$ represents the gain matrix consisting of proportional and derivative components for establishment of system stability. The gain matrix is so computed, that all the eigen-values of the matrix $\mathbf{A}_{\mathrm{c}}=\mathbf{A}_i+\mathbf{B}_i\mathbf{K}$, at all the nominal points over the reference trajectory lies in the left half of the S-plane. This results, $\boldsymbol{\eta}(t)\to\mathbf{0}$ while, $t\to\infty$, which implies asymptotic tracking. The feedback controller influences stability as well as the transient response of the tracking error.

It is also important to note that, the asymptotic tracking performance and stability of the robotic system is achieved with a single static state feedback gain matrix which can be computed offline instead of a variable feedback gain controller. The performance of the variable gain controllers deteriorate with the sampling frequency during gain scheduling.

IV. SIMULATION & RESULTS

The control scheme explained above has been verified by computer simulation utilizing SIMULINK environment of MATLAB. A circular as well as an elliptical path has been chosen as a reference trajectory and subsequent responses have been verified. The kinematic parameters of a real mobile robot developed at laboratory have been used for simulation. The parameters are:

L = 0.694m, b = 0.207m, r = 0.096m, d=0.116m, Δ = 0.05m, m_{P} = 34.648 kg, m_{w} = 1.556 kg (includes the mass of wheel, spur gear, gearbox rotor and motor of each side), N = 91, I_m=0.0354 kg-m^2, I_w=(I_{motor} + I_{gb}) N^2 + ($2I_{sp}$ + I_{wheel}) = 0.0708 kg-m^2, I=I_{P} + $m_{\mathrm{P}}\,\Delta^2$ + $2m_{\mathrm{w}}(\mathrm{b}^2+\mathrm{d}^2)$ + 2 I_m = 5.716 kg-m^2.

The motor parameters have been obtained from the data provided by the manufacturer (maxon motor): K_T = 75.8 mNm/A, K_b = (1/13.1947) rad/V-sec, R_a = 4.75 ohms.

A. Circular Trajectory Tracking

The trajectory definition for circular path is as follows:

$$\mathrm{x_c}=5\,sin(\tfrac{\mathrm{pi*t}}{25}),\ \mathrm{y_c}=5\,cos(\tfrac{\mathrm{pi*t}}{25}),\ \varphi=\tfrac{\mathrm{pi*t}}{25}.$$

To track the circular reference trajectory of various radiuses accurately and to achieve asymptotic stability, the matrices required to design the controller are chosen as,

$$\mathbf{F}=\begin{bmatrix}-17 & 0.8\\ -10 & -12\end{bmatrix},\ \mathbf{Q_0}=\begin{bmatrix}-20 & 15\\ 13 & -20\end{bmatrix},\ \mathbf{R}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

Using these matrices, we obtain $\mathbf{W}$ matrix as,

$$\mathbf{W}=\begin{bmatrix}0.5546 & -0.7150 & -10.000 & 3.0345\\ -0.7150 & 1.4292 & 13.2989 & -10.000\\ -10.000 & 13.2989 & 181.6391 & -59.5862\\ 3.0345 & -10.000 & -59.5862 & 90.6552\end{bmatrix}.$$

The value of the positive scalar γ has been taken as 498.8749.

Using these values, the single static stabilizing controller gain matrix is calculated and given by:

$$\mathbf{K} = \begin{bmatrix} 4240.4 & -199.55 & 249.437 & 0 \\ 2494.4 & 2993.2 & 0 & 249.437 \end{bmatrix}.$$

The circular path tracked by the robot is nothing but the response of the controller and depicted in Fig. 3.

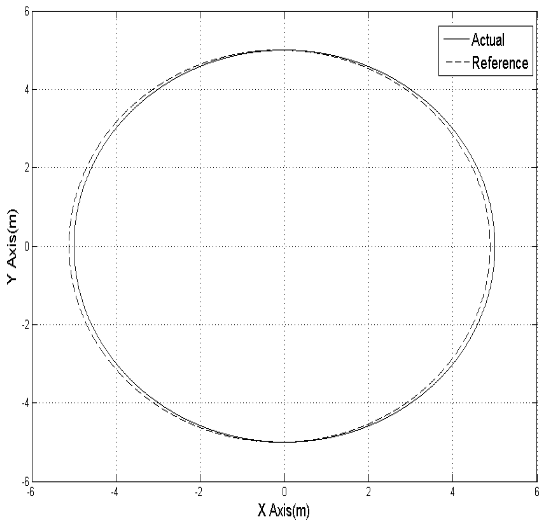

Fig. 3. Actual and Reference Circular Trajectory

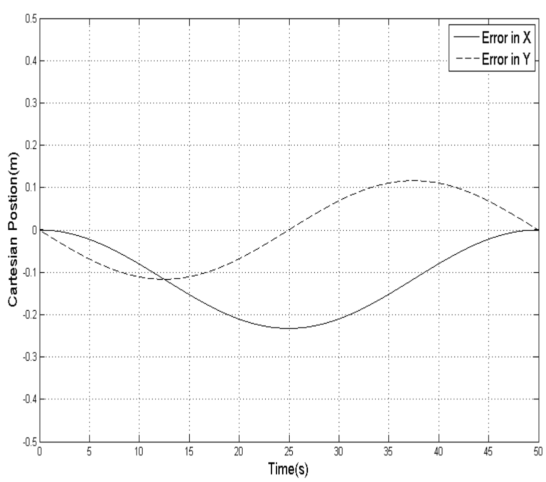

Fig. 4. Error in x and y direction for circular trajectory

The trajectory is followed very accurately as it is obvious from the x, y positional error and tracking error plots as given in Fig. 4 and Fig. 5. The solid line represents x-positional error and dotted line represent y-positional error. The maximum x-positional error is approx. 0.224m and y positional error is approx. 0.115m while tracking a circular trajectory of 5m radius. The maximum distance tracking error can be seen as 0.23m.

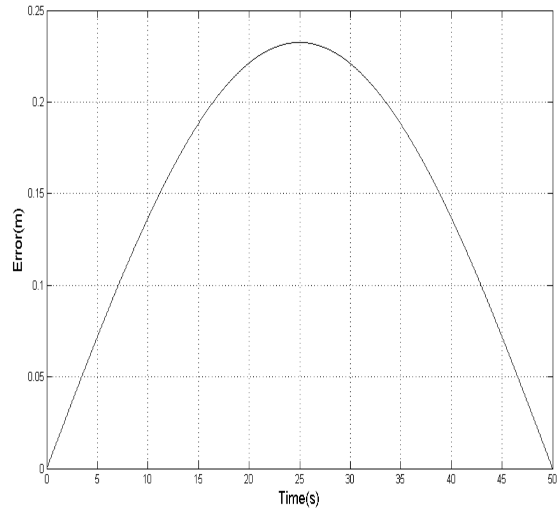

Fig. 5. Total distance tracking error

B. *Elliptical Trajectory Tracking*

The trajectory for the elliptical path is defined as:

$$x_c = 10\,sin(\frac{pi*t}{25}),\, y_c = 15\,cos(\frac{pi*t}{25}),\, \varphi = tan^{-1}\left(\frac{-3\,tan\left(\frac{pi*t}{25}\right)}{2}\right)$$

The values of matrices **F, $\mathbf{Q_0}$, R** & **W** are chosen same as mentioned for the circular path. The positive scalar value γ for this trajectory is calculated as 626.0146. Using these values, the single static stabilizing controller gain matrix is evaluated as:

$$\mathbf{K} = \begin{bmatrix} 5321.1 & -250.40 & 313.01 & 0 \\ 3130.1 & 3756.1 & 0 & 313.01 \end{bmatrix}$$

The trajectory tracked by the robot is represented in Fig. 6.

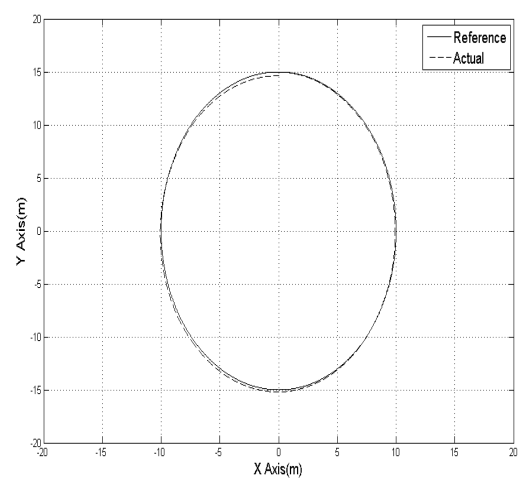

Fig. 6. Actual and Reference Elliptical Trajectory

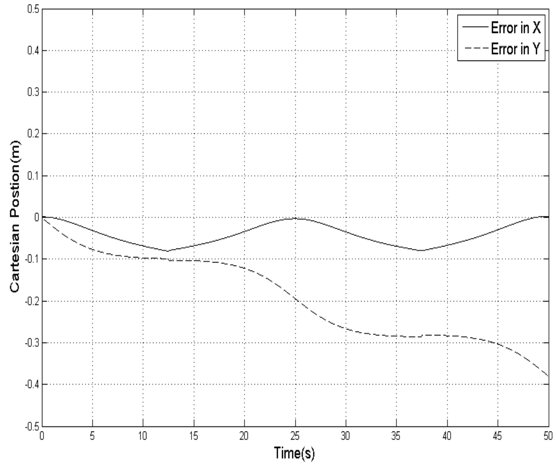

Fig. 7. Error along x & y direction for elliptical trajectory

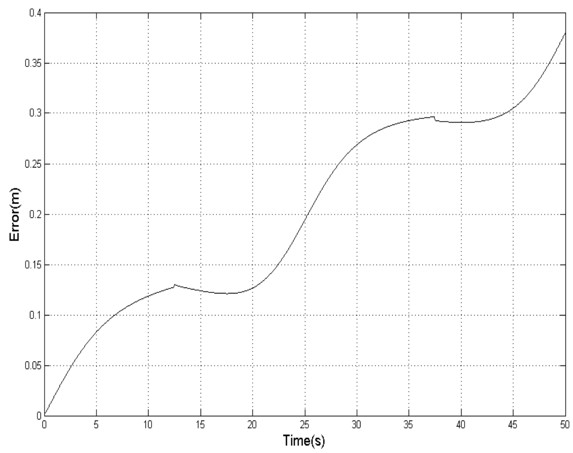

Fig. 8. Total Tracking Error for elliptical trajectory

The results presented through Fig. 7 & Fig. 8 show accurate tracking of the elliptical trajectory. The x & y positional error plot is given in Fig. 7 and the total distance tracking error plot is given in Fig. 8. The maximum x-positional error is 0.0811m and y-positional error is 0.3797m while tracking an elliptical trajectory with transverse diameter as 30m along y-axis and conjugate diameter as 20m along x-axis. The maximum value of the distance tracking error is 0.38m as observed from Fig.8.

V. CONCLUSION

In this endeavor, a simultaneous quadratic stabilization based control scheme for mobile robot control is presented utilizing a novel reduced order dynamic formulation. The actuator dynamics have also been included in the robot dynamics. The dynamic equations are linearized and presented through state space domain to get the structure of block companion form. Reference trajectories are defined and a set of LTI state equations from the linearized dynamics at each nominal point of these trajectories are evaluated. An advanced controller based on Simultaneous Quadratic Stabilization method is envisaged for the mobile robot to achieve asymptotic tracking with stability while following a reference trajectory. The single stabilizing feedback controller is designed utilizing the presented algorithm. Computer simulations have been performed to analyze the effectiveness of the proposed control scheme for different trajectories applied to mobile robot navigation. The simulation results show the precise performance of the controller as verified with the parameters of a real robot. The performance of the controller proves the effectiveness of the proposed method for mobile robot control.

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